New Journal of Physics
The open–access journal for physics

Inseparability inequalities for higher order moments for bipartite systems

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New Journal of Physics **7** (2005) 211
Received 18 July 2005
Published 7 October 2005
Online at http://www.njp.org/
doi:10.1088/1367-2630/7/1/211

Abstract. There are several examples of bipartite entangled states of continuous variables for which the existing criteria for entanglement using the inequalities involving the second-order moments are insufficient. We derive new inequalities involving higher order correlation, for testing entanglement in non-Gaussian states. In this context, we study an example of a non-Gaussian state, which is a bipartite entangled state of the form $\psi(x_a, x_b) \propto (\alpha x_a + \beta x_b)e^{-(x_a^2+x_b^2)/2}$. Our results open up an avenue to search for new inequalities to test entanglement in non-Gaussian states.

The detection and characterization of entanglement in the state of a composite is an important issue in quantum information science. Peres [1] has addressed this issue for the first time to show that inseparability of a bipartite composite system can be understood in terms of negative eigenvalues of partial transpose of its density operator. There are several other criteria for inseparability in terms of correlation entropy and linear entropy [2, 3] and in terms of positivity of Glauber–Sudarshan P-function [4]. However, all these measures cannot be put to experimental tests. To detect entanglement of any composite system experimentally, one needs to have certain criteria in terms of expectation values of some observables.

Using Peres's criterion of separability, Simon [5] has derived certain separability inequalities, violation of which is sufficient to detect entanglement in bipartite systems. These inequalities involve variances of relative position and total momentum coordinates of the two subsystems and thus can be verified experimentally [6]. Duan *et al* [7] have also derived equivalent inequalities independently using the positivity of the quadratic forms. It is further proved that for Gaussian states (states with Gaussian wavefunctions in coordinate space), violation of these

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PII: S1367-2630(05)04010-3

inequalities provides a necessary and sufficient criterion for entanglement. A different form of criterion for entanglement involving second-order moments has been derived by Mancini *et al* [8]. These inequalities have been tested for entangled states produced by optical parametric oscillators and other systems where the output state can be approximated by Gaussian states [9]–[12].

In context of quantum information and communication, non-Gaussian states are equally important as Gaussian states. Several entangled non-Gaussian states have been studied in the literature [13]–[15]. A way to produce non-Gaussian state is via the state reduction method [16]–[18]. Thus, characterization of entanglement in a non-Gaussian state remains an open question. This motivates us to derive new inequalities, when the existing inequalities based on second-order correlation fail to test entanglement in these states.[4] Thus, these inequalities are expected to involve higher order correlation between position and momentum coordinates. It should be borne in mind that there could be non-Gaussian states for which the existing criteria of Simon and Duan *et al* [5, 7] are quite adequate. An example is provided by the pair coherent states of the radiation fields, i.e., states given by [21]

$$\psi = N_0 \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} |n, n\rangle \qquad N_0 = \frac{1}{\sqrt{I_0(2|\zeta|)}}. \tag{1}$$

In this paper, we consider a bipartite entangled state of a bosonic system which, in turn, is a non-Gaussian state in coordinate space. We consider an entangled state for which the existing inseparability inequalities cannot provide any information about the inseparability of the state. We derive new inseparability inequalities to test its entanglement.

We start by deriving the inequalities involving the second-order moments. Consider the set of operators

$$U = \frac{1}{\sqrt{2}}(x_{\mathrm{a}} + x_{\mathrm{b}}), \qquad V = \frac{1}{\sqrt{2}}(p_{\mathrm{a}} + p_{\mathrm{b}}) \quad [U, V] = i. \tag{2}$$

Then, we would have the uncertainty relation

$$\Delta U \Delta V \geqslant \tfrac{1}{2}. \tag{3}$$

We now use Peres–Horodecki criteria of separability in terms of the partial transpose. Under the partial transpose, $x_{\mathrm{b}} \to x_{\mathrm{b}}$, $p_{\mathrm{b}} \to -p_{\mathrm{b}}$. Hence the condition that the partial transpose of a density matrix is also a genuine density matrix would imply that

$$\Delta \left(\frac{x_{\mathrm{a}} + x_{\mathrm{b}}}{\sqrt{2}} \right) \Delta \left(\frac{p_{\mathrm{a}} - p_{\mathrm{b}}}{\sqrt{2}} \right) \geqslant \frac{1}{2}. \tag{4}$$

This inequality was first derived by Mancini *et al* [8] using a very different method.[5] Thus if a bipartite system is separable then (4) should be obeyed. Violation of equation (4) gives a sufficient

[4] Certain results of a general nature have been proved. For example, Eisert *et al* [19] and Clifton and Halvorson [20] have proved that pure states with infinite entropy of entanglement form a trace norm dense set. However, the cases we examine correspond to finite entropy of entanglement.

[5] Earlier Simon used a similar argument to derive his inequalities.

condition for entanglement. The inequality of Duan *et al* follow from (4) by using the relations

$$\begin{aligned} M^2 &= M_-^2 + 4M_x, \\ M &= [\langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle], \\ M_- &= [\langle(\Delta u)^2\rangle - \langle(\Delta v)^2\rangle], \\ M_x &= \langle(\Delta u)^2\rangle\langle(\Delta v)^2\rangle, \end{aligned} \tag{5}$$

where

$$u = x_a + x_b \quad \text{and} \quad v = p_a - p_b. \tag{6}$$

From equation (5) it is clear that if the inequality $M_x \geqslant 1$ (which is the separability criterion of Mancini *et al*) holds, then the criterion $M \geqslant 2$ which is the separability criterion of Duan *et al*, is automatically satisfied for all values of M_-. But if $M_x < 1$, then nothing can be said about the exact value of M. It can be greater than or less than 2 depending on the values of M_x and M_-. The above analysis implies that the separability criterion given by Mancini *et al* and that given by Duan *et al* are interrelated with each other. Furthermore, $M_x < 1$ is stronger than the criterion $M < 2$ for inseparability. This follows from $M_x \leqslant M/2$. We also note that Duan *et al* derived a more general separability inequality

$$\left|m^2 - \frac{1}{m^2}\right| \leqslant M < m^2 + \frac{1}{m^2}, \tag{7}$$

where

$$u = |m|x_a + \frac{1}{m}x_b, \qquad v = |m|p_a - \frac{1}{m}p_b. \tag{8}$$

For bipartite Gaussian states, the inequalities for second-order correlations are also sufficient. Equivalent necessary and sufficient conditions for separability of Gaussian states have been derived by Englert and Wodkiewicz [22] using a density operator formalism. They have shown that the positivity of the partial transposition and P-representability of the separable Gaussian states are closely related.

In this paper, we focus on the following bipartite continuous variable Bell state formed from ground and excited states of the harmonic oscillators

$$\psi(x_a, x_b) = \sqrt{\frac{2}{\pi}}(\alpha x_a + \beta x_b)\mathrm{e}^{-(x_a^2+x_b^2)/2}, \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{9}$$

which is the state of a composite system of bosonic particles. It clearly represents a non-Gaussian state in coordinate space. The non-classical properties of such states were studied in [13]. A recent experimental proposal discusses how to generate non-Gaussian states by subtracting a photon from each mode of a two-mode squeezed vacuum state [15].

The Peres–Horodecki criterion [1] is known to be necessary and sufficient for inseparability for bipartite systems in (2×2) and (2×3) dimensions, but to be only sufficient for any higher dimensions. This criterion states that if the partial transpose of a bipartite density matrix has at least one negative eigenvalue, then the state must be inseparable. Next, we apply this criterion to test the inseparability of the state (9). The density matrix of this state is given by

$$\rho = |\psi\rangle\langle\psi| = |\alpha|^2|1,0\rangle\langle 1,0| + |\beta|^2|0,1\rangle\langle 0,1| + (\alpha^*\beta|0,1\rangle\langle 1,0| + \text{H.c.}), \quad (10)$$

where

$$|1,0\rangle \equiv \sqrt{\frac{2}{\pi}}x_{\mathrm{a}}\mathrm{e}^{-(x_{\mathrm{a}}^2+x_{\mathrm{b}}^2)/2}. \quad (11)$$

Taking the partial transpose of the second subsystem, we obtain the following density matrix

$$\rho^{\mathrm{PT}} = |\alpha|^2|1,0\rangle\langle 1,0| + |\beta|^2|0,1\rangle\langle 0,1| + (\alpha^*\beta|0,0\rangle\langle 1,1| + \text{H.c.}). \quad (12)$$

The four eigenvalues of the above density matrix can be calculated as $|\alpha|^2$, $|\beta|^2$ and $\pm|\alpha||\beta|$. Clearly, the negative eigenvalue of ρ^{PT} confirms the inseparability of the state (9) under consideration.

Now, we examine the validity of the existing inseparability inequalities (7) and (4) for the entangled state (9). For the conjugate variables u and v defined by equation (8), we find that

$$\langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle = |m|^2 + \frac{1}{m^2} + 2\left(|\alpha|^2|m|^2 + \frac{1}{m^2}|\beta|^2\right), \quad (13)$$

which is clearly greater than $|m|^2 + 1/m^2$. Thus, though the state (9) is entangled, the criterion (7) cannot exploit this fact. In other words, violation of the criterion (7), as shown above, would conclude that the state under consideration is separable, which definitely is not the case. We further find that for $m = 1$

$$\langle(\Delta u)^2\rangle\langle(\Delta v)^2\rangle = 4 - (\alpha\beta^* + \alpha^*\beta)^2 = 4 - 4[\mathrm{Re}(\alpha\beta^*)]^2, \quad (14)$$

which has minimum value equal to 3, which implies that $\langle(\Delta u)^2\rangle\langle(\Delta v)^2\rangle$ is always greater than unity for $m = 1$. According to the inequality (4), this refers to separability in the state which is again not the case. From the above discussion, we conclude that the existing inseparability criteria based on second-order correlations do not provide correct information about the inseparability of a standard bipartite entangled state which in turn is non-Gaussian. This warrants a search for new inequalities involving higher order correlations, to test the inseparability of such states.

Needless to say that since there is an infinity of these higher order correlations, one could construct a very large number of such inequalities involving higher order correlations. In what follows, we consider the next logical correlations. Our analysis below is reminiscent of what has been done in context of non-classical light [23, 24]. We could consider the following set of operators:

$$S_{\mathrm{x}} = \frac{a^\dagger b + ab^\dagger}{2}, \qquad S_{\mathrm{y}} = \frac{a^\dagger b - ab^\dagger}{2\mathrm{i}}, \qquad S_{\mathrm{z}} = \frac{a^\dagger a - b^\dagger b}{2}. \quad (15)$$

The operators S_i obey the algebra of angular momentum operators and hence the uncertainty relation $\Delta S_x \Delta S_y \geqslant \frac{1}{2}|\langle S_z\rangle|$ would give, for example

$$\Delta\left[\frac{a^\dagger b + ab^\dagger}{2}\right]\Delta\left[\frac{a^\dagger b - ab^\dagger}{2i}\right] \geqslant \frac{1}{2}\left|\left\langle\frac{a^\dagger a - b^\dagger b}{2}\right\rangle\right|. \tag{16}$$

It is known that under partial transpose, a separable density matrix remains as a valid density operator. Using this property of partial transpose, we expect that for a separable state, the following inequality is also valid

$$[\langle a^\dagger abb^\dagger\rangle + \langle aa^\dagger b^\dagger b\rangle + \langle a^{\dagger 2} b^{\dagger 2}\rangle + \langle a^2 b^2\rangle - \langle a^\dagger b^\dagger + ab\rangle^2] \\ \times[\langle a^\dagger abb^\dagger\rangle + \langle aa^\dagger b^\dagger b\rangle - \langle a^{\dagger 2} b^{\dagger 2}\rangle - \langle a^2 b^2\rangle + \langle a^\dagger b^\dagger - ab\rangle^2] \geqslant |\langle a^\dagger a - b^\dagger b\rangle|^2, \tag{17}$$

which has been obtained from equation (16) under the partial transpose $b \leftrightarrow b^\dagger$. A violation of (17) would imply that the state is entangled. However, for the state (9), $|(|\alpha|^2 - |\beta|^2)| \leqslant 1$. Thus, the inequality (17) is not violated and hence does not lead to any new information regarding the inseparability of the state. Next, we consider the following operators satisfying SU(1,1) algebra

$$K_x = \frac{a^\dagger b^\dagger + ab}{2}, \qquad K_y = \frac{a^\dagger b^\dagger - ab}{2i}, \qquad K_z = \frac{a^\dagger a + b^\dagger b + 1}{2}. \tag{18}$$

Such operators previously have been used in consideration of higher order squeezing [23]. The uncertainty inequality would give

$$\Delta\left[\frac{a^\dagger b^\dagger + ab}{2}\right]\Delta\left[\frac{a^\dagger b^\dagger - ab}{2i}\right] \geqslant \frac{1}{2}\left|\left\langle\frac{a^\dagger a + b^\dagger b + 1}{2}\right\rangle\right|. \tag{19}$$

Using the partial transpose as above, we get a new inequality for separability

$$[\langle a^\dagger ab^\dagger b\rangle + \langle aa^\dagger bb^\dagger\rangle + \langle a^{\dagger 2} b^2\rangle + \langle a^2 b^{\dagger 2}\rangle - \langle a^\dagger b + ab^\dagger\rangle^2] \\ \times[\langle a^\dagger ab^\dagger b\rangle + \langle aa^\dagger bb^\dagger\rangle - \langle a^{\dagger 2} b^2\rangle - \langle a^2 b^{\dagger 2}\rangle + \langle a^\dagger b - ab^\dagger\rangle^2] \geqslant |\langle a^\dagger a + bb^\dagger\rangle|^2. \tag{20}$$

For the state (9), the above relation leads to the following result

$$|\alpha^*\beta|^2 - 2[\mathrm{Re}(\alpha^*\beta)]^2[\mathrm{Im}(\alpha^*\beta)]^2 \leqslant 0, \tag{21}$$

which is always violated for all values of α and β. Thus, the state under consideration is inseparable according to this inequality (20) which is in conformity with the Peres–Horodecki criterion. This inequality, which is based on higher order correlation, is thus successful to test inseparability in the non-Gaussian states like (9), while the existing inequalities based on second-order correlation fail to do so. This result opens up an avenue to search for general inseparability inequalities for non-Gaussian states.

Note that the inequality (20) can also be expressed in terms of position and momentum variables of the two subsystems as

$$[\Delta^2(x_a x_b) + \Delta^2(p_a p_b) + \langle x_a p_a p_b x_b\rangle + \langle p_a x_a x_b p_b\rangle - 2\langle x_a x_b\rangle\langle p_a p_b\rangle] \\ \times[\Delta^2(x_a p_b) + \Delta^2(p_a x_b) - \langle x_a p_a x_b p_b\rangle - \langle p_a x_a p_b x_b\rangle + 2\langle x_a p_b\rangle\langle p_a x_b\rangle] \\ \geqslant \tfrac{1}{4}|\langle x_a^2\rangle + \langle p_a^2\rangle + \langle x_b^2\rangle + \langle p_b^2\rangle|^2. \tag{22}$$

In order to detect entanglement using equation (20), we need to make a variety of homodyne measurements [25]–[28]. Such measurements would yield the distribution of quadratures.

In conclusion, we have shown in context of a bosonic non-Gaussian state of the Bell form, that the existing inseparability inequalities based on second-order correlations are not enough to test the entanglement. We have derived a new set of separability inequalities using the Peres–Horodecki criterion of separability. This new inequality involves higher order correlation of quadrature variables and can be tested experimentally as discussed above. Violation of this inequality detects entanglement in the non-Gaussian state under consideration. The failure of the existing criteria in terms of the second-order moments is perhaps a reflection of the fact that the state (9) in no limit goes over to a Gaussian state.

Acknowledgments

Support from NSF grant no. CCF-0524673 is gratefully acknowledged.

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